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Upstream influence in a two-fluid system

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Benjamin (1970) has calculated the upstream influence in open-channel flow, and has argued that a similar effect occurs when a body is moved along the axis of a tube of uniformly rotating fluid. In the present paper Benjamin's work is extended to the case of interfacial waves in a two-fluid system. It is shown that there are certain special flows for which the upstream influence vanishes.

1. Introduction

In its simplest terms, the question to be answered is: what happens when a small body, or weak dipole, starts to move horizontally from rest, and continues at a subcritical velocity through a two-fluid system that is stably stratified and initially undisturbed? (A subcritical velocity is one which is less than the propagation speed of long waves, as predicted by linear theory.) Two related phenomena

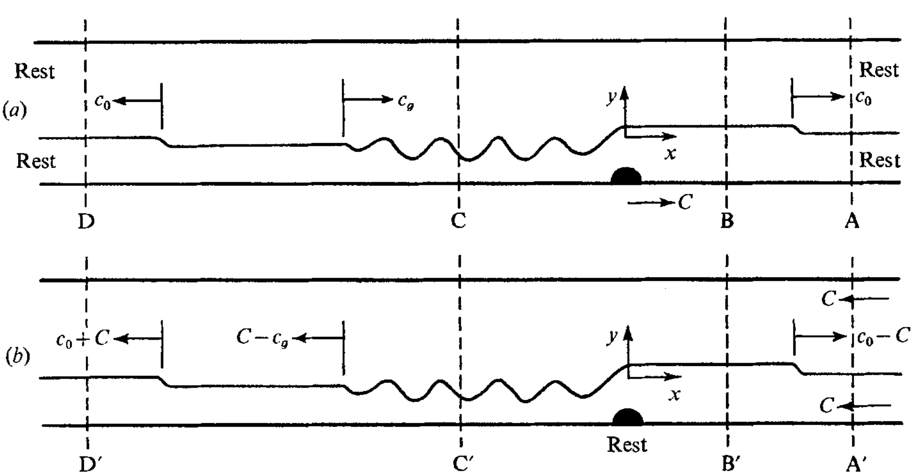

FIGURE 1. Illustration of the interfacial-wave problem: (*a*) obstacle propelled at constant velocity C in water initially at rest; (*b*) obstacle fixed in stream approaching with velocity C.

arise: the body experiences a drag due to the continual development of (small amplitude) waves on the leeward side, and surges propagate upstream and downstream at the long-wave speed. In particular, the flow remains undisturbed only ahead of the ever-lengthening upstream surge (figure 1). By taking the limit $t \to \infty$ at any distance x from the body, we obtain a steady flow $\mathscr{F}_s$. Therefore, while it must be true that the motion tends to its undisturbed form $\mathscr{F}_0$ as $|x| \to \infty$ for any fixed t, no matter how large, we cannot infer that in the steady flow $\mathscr{F}_s$ the flow far upstream $\mathscr{F}_{s,+}$ will be the same as the original undisturbed flow. That is,

$$\mathscr{F}_{s,+} = \lim_{x\to\infty} \lim_{t\to\infty} \mathscr{F} \neq \lim_{t\to\infty} \lim_{x\to\infty} \mathscr{F} = \mathscr{F}_0.$$

Indeed we shall see that the flow $\mathscr{F}_{s,+}$ differs from the original undisturbed flow $\mathscr{F}_0$ by second-order changes in the height of the interface, and by a shear across the interface which is of the same order as the Stokes drift velocity in the wave train.

These upstream effects should be distinguished from the stronger effect of blocking, which is sometimes represented by a model based on non-dispersive non-linear shallow-water theory of the type suggested by figure 2 (Long 1970).

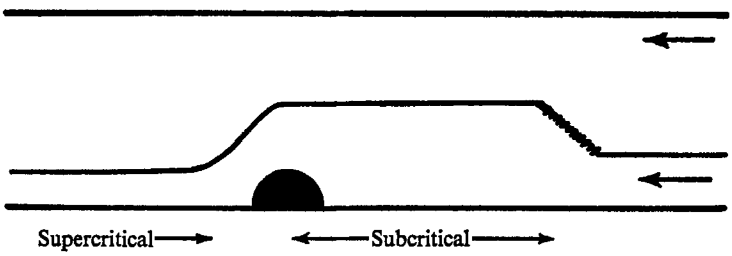

FIGURE 2. A model of 'blocking' when an obstacle is introduced into a stream (Long 1970, figure 3).

In the present paper, the main approach is similar to that of Benjamin (1970), but it does differ in certain respects. Benjamin completed the calculation of upstream influence for open-channel flow and then gave an argument for the necessity of a similar effect when a body is moved along the axis of a tube of rotating fluid. Explicit details of the amount of upstream influence were available only for open-channel flow. In this paper we shall calculate the amount of upstream influence for a two-fluid system.

The calculations depend on the use of conservation equations

$$\frac{\partial P_j}{\partial t} + \frac{\partial Q_j}{\partial x} = 0 \tag{1.1}$$

for suitable 'densities' P_j and 'fluxes' Q_j. A suitable set of conservation equations is given in §2. The equations for conservation of mass and energy are straightforward, but need to be supplemented by one further equation. To this end an extension to general stratified flows of Benjamin's impulse principle was originally used, but we shall present, and use, a different conservation equation, (2.8), more suited to the problem at hand. Finally in §2, expressions are given for the densities P_j and fluxes Q_j appropriate to uniform wave trains, and, with further approximation to slowly varying wave trains. For surface waves, the conservation equations were presented in this form by Whitham (1962).

In §3.1 we apply the conservation equations to the virtually steady-flow region between B and C (figure 1) to find the wave resistance and the change in mean level of the interface across the obstacle. In § 3.2, by examining the whole flow system between A and D (figure 1), we calculate the amount of upstream influence. Simplifications, including the Boussinesq approximation, are briefly discussed. The analysis brings to light some special flows without upstream influence.

The analysis cannot, however, be said to be rigorous. It has not been proved that the flow near the body becomes steady, and there are still some doubts about the perturbation procedure.

The problem can be viewed as a perturbation problem involving two parameters: large time and small amplitude. The known results of the linearized theory of surface waves are, first, that transients in the neighbourhood of the obstacle decay like $t^{-\frac{1}{2}}$, and, second, that transients at the rear end of the wave train are confined to a region whose length is $O(t^{\frac{1}{2}})$, the overall length of the train being $O(t)$. It is also known that a long wave such as the forward surge can have an oscillatory frontal region, but the length of this is $O(t^{\frac{1}{3}})$. Similar order estimates hold for interfacial waves; that is, the transient regions have lengths $O(t^{\nu})$, $\nu < 1$. An example of the sort of differences which arise in the estimates is if one of the layers is infinitely deep the length of the oscillatory frontal region is $O(t^{\frac{1}{2}})$.

We shall assume, in accord with the linearized theory, that after a long time the motion in the neighbourhood of the obstacle becomes steady.

If we average over distances $O(t^{\beta})$, large compared to transient regions but small compared to the length of the wave train (that is, $0 < \beta_0 < \beta < 1$, for suitable β_0), we shall obliterate the transient zones by the averaging process. Benjamin formalized this by means of an averaging operator A defined as follows:

$$\bar{\zeta}(x,y,t) \equiv A\zeta = \frac{1}{2\xi}\int_{x-\xi}^{x+\xi} \zeta(X,y,t)\,dX,$$

in which $\xi = \alpha t^{\beta}$ with $\beta_0 < \beta < 1$. We then have

$$A\left(\frac{\partial \zeta}{\partial x}\right) = \frac{\partial \bar{\zeta}}{\partial x},$$

and

$$A\left(\frac{\partial \zeta}{\partial t}\right) = \frac{\partial \bar{\zeta}}{\partial t} + \frac{\beta\bar{\zeta}}{t} - \frac{\beta}{2t}\{\zeta(x+\xi,y,t)+\zeta(x-\xi,y,t)\}$$

$$\sim \frac{\partial \bar{\zeta}}{\partial t} \quad \text{for} \quad t \to \infty.$$

This can be used to replace the exact conservation equations (1.1) by their averages,

$$\frac{\partial \bar{P}_j}{\partial t} + \frac{\partial \bar{Q}_j}{\partial x} = 0. \tag{1.2}$$

The simplification due to small amplitude then leads, in the case of surface gravity waves, to Whitham's (1962) system obtained there in the slightly different context of slowly varying wave trains of small amplitude.

It would be desirable, but difficult, to put the perturbation procedure on a firmer basis. The averaged conservation equations are clearly valid *locally* to the order calculated by Benjamin, but we apply them over large stretches of the flow domain. It is possible to show the theory is valid for t large and wave amplitude a so small that ta is still small. The argument shows that for ta not small the terms in $\bar{P}_j$, $\bar{Q}_j$ of higher order in a than the second (the errors in our approximation) may accumulate over the large regions throughout which the conservation laws are applied. The validity of the theory for very large time is crucial, but as yet no convincing demonstration of it has been given.

These remarks on the reliability of the perturbation procedure apply not only to the two-fluid problem but also to the flows considered by Benjamin. The two-layer model, however, introduces a further difficulty. Although the linearized initial-value problem is well posed, it is not at all obvious that the same is true of the non-linear problem. The possibility of Kelvin–Helmholtz instability must be considered whenever even the smallest shear arises across any portion of the interface. The non-linear problem is well set if we have some additional effect, such as small viscosity or surface tension, or some density structure instead of a sharp interface.

2. The conservation equations

We shall find an approximate solution to the initial-value problem in which a body moves from rest in a two-fluid system. We suppose that the density of the lower layer is ρ_1, and that its undisturbed depth is d_1. The upper layer has density $\rho_2(<\rho_1)$, and depth d_2. Take $y = 0$ as the undisturbed level of the interface, and describe the height of the interface at subsequent times by $y = \eta(x,t)$. Initially the fluid is at rest; at $t = 0$ a small body starts to move along the bottom. By Kelvin's circulation theorem (for an inviscid liquid of uniform density), the flow in both layers is irrotational. We have the usual kinematical and dynamical conditions at the interface: the interface is a material surface, and the pressure is continuous across it. Thus, with the notation $\mathbf{u} = (u,v) = \nabla\phi$,

$$\left.\begin{aligned}\nabla^2\phi_1 &= 0 \quad (-d_1 < y < \eta),\\ \nabla^2\phi_2 &= 0 \quad (\eta < y < d_2),\end{aligned}\right\} \tag{2.1}$$

$$\eta_t + u_i\eta_x = v_i \quad \text{at} \quad y = \eta(x,t), \tag{2.2}$$

$$[\rho(\partial\phi(x,y,t)/\partial t + \tfrac{1}{2}(u^2+v^2)+gy)] = \chi(t) \quad \text{at} \quad y = \eta(x,t), \tag{2.3}$$

where $[f] = f_1 - f_2$ and $i = 1, 2$ distinguishes the two fluids. Further, on horizontal boundaries,

$$v = 0. \tag{2.4}$$

If the velocity of the body is $\mathbf{U}(t)$ in the x direction, we have

$$(\mathbf{u} - \mathbf{U}(t)).\mathbf{n} = 0 \tag{2.5}$$

at the body surface, where $\mathbf{n}$ is the normal to the body. Usually, we shall suppose that the body is accelerated instantaneously from rest to a constant velocity C. In what follows, velocities u_i, v_i with no position argument indicated denote values at the interface, or, more precisely, limiting values as the interface is approached.

Integrating equations (2.1) with respect to y across each layer, and applying the kinematical boundary conditions (2.2), we obtain

$$\frac{\partial \eta}{\partial t}+\frac{\partial Q_1}{\partial x} = 0, \quad \text{where} \quad Q_1 = \int_{-d_1}^{\eta} u_1(y)\,dy, \tag{2.6}$$

$$-\frac{\partial \eta}{\partial t}+\frac{\partial Q_2}{\partial x} = 0, \quad \text{where} \quad Q_2 = \int_{\eta}^{d_2} u_2(y)\,dy. \tag{2.7}$$

We obtain a further conservation equation as follows: substituting from

$$\frac{\partial}{\partial t}\phi_i(x,\eta(x,t)t) = \frac{\partial}{\partial t}\phi_i(x,y,t)+\eta_t v(x,y,t), \quad \text{at} \quad y = \eta(x,t),$$

into the pressure continuity condition (2.3), and then differentiating with respect to x (at fixed t and with $y = \eta(x,t)$), we have

$$\frac{\partial}{\partial x}\frac{\partial}{\partial t}[\rho\phi(x,\eta,t)]+\frac{\partial}{\partial x}[\rho(-\eta_t v+\tfrac{1}{2}(u^2+v^2)+g\eta)] = 0.$$

Since
$$\frac{\partial}{\partial x}\frac{\partial}{\partial t}\phi_i(x,\eta(x,t),t) = \frac{\partial}{\partial t}(u_i+\eta_x v_i),$$

we have
$$\frac{\partial}{\partial t}[\rho(u+\eta_x v)]+\frac{\partial}{\partial x}[\rho(-\eta_t v+\tfrac{1}{2}(u^2+v^2)+g\eta)] = 0.$$

Using the kinematical conditions, we finally have

$$\left.\begin{aligned} \frac{\partial I}{\partial t}+\frac{\partial J}{\partial x} &= 0 \\ I &= [\rho(u+\eta_x v)], \\ J &= [\rho(\eta_x uv+\tfrac{1}{2}(u^2-v^2)+g\eta)] \end{aligned}\right\} \tag{2.8}$$

with

For steady flow, this equation states that the difference $[\rho(\frac{1}{2}(u^2+v^2)+g\eta)]$ in the Bernoulli constants across the interface is constant.

Let us consider the approximation for large time. By means of the averaging operator defined in §1, equations (2.6)–(2.8) become

$$\left.\begin{aligned} \frac{\partial \bar{\eta}}{\partial t}+\frac{\partial \bar{Q}_1}{\partial x} &= 0, \\ -\frac{\partial \bar{\eta}}{\partial t}+\frac{\partial \bar{Q}_2}{\partial x} &= 0, \\ \frac{\partial \bar{I}}{\partial t}+\frac{\partial \bar{J}}{\partial x} &= 0. \end{aligned}\right\} \tag{2.9}$$

The next approximation is to suppose that the displacement of the interface is small. In view of the approximation for large time it is plausible to regard the flow as a slowly varying wave train. Consider waves of frequency $\omega(k)$, wave-number k on a stream with mean speeds $\mathscr{U}_1, \mathscr{U}_2$ and undisturbed depths d_1, d_2. With the notation $\omega_i^+ = \omega-\mathscr{U}_i k$, the linearized dispersion relation is

$$f_1+f_2 = 1, \quad \text{where} \quad f_i = \frac{\rho_i(\omega_i^+)^2}{g(\rho_1-\rho_2)\,k\tanh kd_i} \tag{2.10}$$

(Lamb 1932, §§232–4); this may be regarded as giving ω as a function of k. We note for future reference a result obtained by differentiating (2.10) with respect to k:

$$-\left\{d_1\frac{\partial f_1}{\partial d_1}+d_2\frac{\partial f_2}{\partial d_2}\right\}=\left\{\frac{2c_{g1}^+}{c_1^+}f_1+\frac{2c_{g2}^+}{c_2^+}f_2-1\right\}, \tag{2.11}$$

where $c_i^+=\omega_i^+/k,\quad c_{gi}^+=d\omega_i^+/dk.$

We compute the densities and fluxes for a uniform wave train from the quantities in table 1. The amplitude of the waves is a, and the wave energy per unit area is therefore $E_w=\frac{1}{2}(\rho_1-\rho_2)ga^2$. Introducing the notation

$$\bar{\eta}=b,\quad h_1=d_1+b\quad\text{and}\quad h_2=d_2-b$$

and averaging over a wavelength, we obtain the following expressions for the averaged densities and fluxes in (2.9):

$$\left.\begin{aligned}\bar{Q}_i&=\mathscr{U}_ih_i+\frac{E_wf_i}{\rho_ic_i^+},\\ \bar{I}&=[\rho\mathscr{U}],\\ \bar{J}&=[\tfrac{1}{2}\rho\mathscr{U}^2+\rho gb-\tfrac{1}{2}E_w\partial f/\partial d].\end{aligned}\right\} \tag{2.12}$$

The error of these expressions is $o(E_w)$.

Velocity in the upper layer:	$u_2(x,y,t)=\mathscr{U}_2-a\omega_2^+\dfrac{\cosh k(d_2-y)}{\sinh kd_2}\cos\theta$
	$v_2(x,y,t)=a\omega_2^+\dfrac{\sinh k(d_2-y)}{\sinh kd_2}\sin\theta$
Displacement of the interface:	$\eta(x,t)=b+a\cos\theta;\ \theta=kx-\omega t$
Velocity in the lower layer:	$u_1(x,y,t)=\mathscr{U}_1+a\omega_1^+\dfrac{\cosh k(d_1+y)}{\sinh kd_1}\cos\theta$
	$v_1(x,y,t)=a\omega_1^+\dfrac{\sinh k(d_1+y)}{\sinh kd_1}\sin\theta$

TABLE 1. Formulae for velocities u_i, v_i used in computing averaged densities and fluxes

The assumption that both layers are initially at rest and of uniform depths d_1, d_2 means that the velocities $\mathscr{U}_i$ and height changes b are due entirely to the wave motion, and are all $O(E_w)$. We now linearize the conservation equations, retaining only terms of the first order in E_w, $\partial k/\partial x$, and $\partial k/\partial t$. Defining the mass-transport velocities

$$U_i=\mathscr{U}_i+\frac{E_wf_i}{\rho_ic_id_i},$$

we have

$$\frac{\partial b}{\partial t}+d_1\frac{\partial U_1}{\partial x}=0, \tag{2.13}$$

$$-\frac{\partial b}{\partial t}+d_2\frac{\partial U_2}{\partial x}=0, \tag{2.14}$$

and
$$\frac{\partial}{\partial t}[\rho\mathscr{U}]+\frac{\partial}{\partial x}\left[\rho g b-\tfrac{1}{2}E_w\frac{\partial f}{\partial d}\right]=0, \tag{2.15a}$$

or
$$\frac{\partial}{\partial t}\left[\rho U-\frac{E_w f}{cd}\right]+\frac{\partial}{\partial x}\left[\rho g b-\tfrac{1}{2}E_w\frac{\partial f}{\partial d}\right]=0. \tag{2.15b}$$

Away from any agency that does work,

$$\frac{\partial E_w}{\partial t}+c_g\frac{\partial E_w}{\partial x}=0. \tag{2.16a}$$

Further, changes in the wave-number propagate at the group velocity

$$\frac{\partial k}{\partial t}+c_g\frac{\partial k}{\partial x}=0. \tag{2.17}$$

(If $\rho_2=0$, this is equivalent to the set of equations given by Whitham 1962.) The system is hyperbolic, and one characteristic speed is clearly the group velocity. Noting that the energy equation, (2.16*a*), uncouples from the first three, (2.13)–(2.15), we can most easily obtain the remaining characteristic velocities by treating the energy (E_w) terms of (2.15*b*) as forcing terms $f(x-c_g t)$. By standard procedures we find that one is infinite, and the other two are $+c_0$ and $-c_0$, where

$$c_0^2=g(\rho_1-\rho_2)\bigg/\left(\frac{\rho_1}{d_1}+\frac{\rho_2}{d_2}\right)$$

is the long-wave speed according to linearized theory. One characteristic velocity is infinite because changes in total volume flux at any point cause the volume flux everywhere else to change instantaneously.

Equation (2.16*a*) must be modified if work is being done on the fluid. Because of the averaging, the effects of the body can be included just as if they were concentrated at a point on the x axis; thus, they appear as a forcing term in the energy equation,

$$\frac{\partial E_w}{\partial t}+c_g\frac{\partial E_w}{\partial x}=\mathscr{D}C\delta(x-Ct), \tag{2.16b}$$

since the rate at which the body does work is the drag $\mathscr{D}$ times its speed C.

We are to solve the system (2.13), (2.14), (2.15*b*), (2.16*b*) and (2.17) with zero initial data. Using the fact that discontinuities propagate along characteristics, we anticipate the following qualitative features of the solution (see figure 1). (i) The forward surge advances at the velocity c_0, thus drawing ahead of the obstacle moving at subcritical velocity C. (Dispersive effects present at the surge are obliterated by the averaging, as discussed in §1.) (ii) The oscillatory wave train is stationary relative to the obstacle, so that the phase velocity c is equal to C, and the downstream end of the wave train advances at velocity $c_g<c$, thus falling steadily behind the obstacle. (Transients at the back are, again, obliterated by the averaging.) (iii) The rearward surge recedes at the velocity $-c_0$. In §3 the details of the solution are completed. We apply the conservation equations,

$$\frac{\partial P}{\partial t}+\frac{\partial Q}{\partial x}=A(t)\,\delta(x-Vt)$$

in the form of jump conditions. Integration over discontinuities gives

$$[Q]_{x=Vt-}^{x=Vt+} = V[P]_{x=Vt-}^{x=Vt+} + A(t),$$

where V is the velocity of the jump.

The values b of the mean height changes in the forward surge, the oscillatory train and the rearward surge, will be denoted by δ_+, δ_w and δ_- respectively. The mass transport velocities in the corresponding regions are all $O(E_w)$ like the above δ; they will be denoted by U_{i+}, U_{iw} and U_{i-}.

3. Main calculation: no shear across the interface

3.1. *The classical argument for the wave resistance*

We apply the conservation equations to the region between B and C in figure 1(a). As in Lamb (1932, §249) energy considerations give the classical formula for the wave resistance,

$$\mathscr{D}c = (c-c_g)\,E_w, \tag{3.1}$$

which states that the rate at which the body does work is equal to the rate of increase of energy in the uniform wave train. The remaining conservation equations are

$$(\delta_w-\delta_+)c = (U_{1w}-U_{1+})\,d_1,$$

$$-(\delta_w-\delta_+)c = (U_{2w}-U_{2+})\,d_2,$$

$$\left[\rho(U_w-U_+)-\frac{E_w f}{cd}\right]c = g(\rho_1-\rho_2)(\delta_w-\delta_+)-\tfrac{1}{2}E_w\left[\frac{\partial f}{\partial d}\right].$$

Eliminating the velocities $U_{iw}-U_{i+}$ we have

$$\delta_+-\delta_w = \frac{E_w}{g(\rho_1-\rho_2)(1-c^2/c_0^2))}\left[\frac{f}{d}-\frac{1}{2}\frac{\partial f}{\partial d}\right]. \tag{3.2}$$

It appears that whether the mean level of the interface is raised or lowered generally depends, not only on ρ_1, ρ_2, d_1, d_2, but also on the speed at which the obstacle is propelled through the fluid. The mean level of the interface is lowered ($\delta_+-\delta_w > 0$) or raised ($\delta_+-\delta_w < 0$), according as

$$\left[\frac{f_1}{d_1}-\frac{1}{2}\frac{\partial f_1}{\partial d_1}\right]\Bigg/\left[\frac{f_2}{d_2}-\frac{1}{2}\frac{\partial f_2}{\partial d_2}\right] \gtrless 1. \tag{3.3a}$$

(In the free surface problem, the mean level is always lowered.) This condition can be rewritten as

$$\frac{\rho_1 F(kd_1)}{\rho_2 F(kd_2)} \gtrless 1, \tag{3.3b}$$

where
$$F(x) = \frac{\coth x}{x}-\tfrac{1}{2}\operatorname{cosech}^2 x,$$

so that $\quad F(x) \sim 1/2x^2 \quad$ as $\quad x\to 0 \quad$ and $\quad F(x) \sim 1/x \quad$ as $\quad x\to\infty$.

Also, F is a monotonically decreasing function. Thus for a Boussinesq fluid, we see that the mean level of the interface is lowered behind the obstacle if $d_1 < d_2$, and raised if $d_1 > d_2$.

The criterion (3.3*b*) simplifies in the limit $k \to 0$, although $a/k^2d_1^3$ and $a/k^2d_2^3$ must remain small for validity of the theory. Note also that since $c \to c_0$ and $k \to 0$ the denominator of (3.2) tends to zero. The level is then lowered or raised behind the obstacle, relative to the level in front, according as

$$\rho_1/\rho_2 \gtrless d_1^2/d_2^2.$$

In a different context (Benjamin 1966), this criterion determines whether (*a*) the crests of cnoidal waves are narrower than their troughs and the solitary wave is a wave of elevation (as happens in surface waves), or (*b*) the crests are broader than the troughs and the solitary wave is a wave of depression.

One can consider various other limits. The limit $k \to \infty$ is probably less physically interesting than the limit $k \to 0$, because the slightest shear makes the lee waves unstable. In this case, the condition (3.3*b*) reduces to

$$\rho_1/\rho_2 \gtrless d_1/d_2.$$

A different limiting situation arises if one of the fluid depths is very large: then the mean level is raised ($\delta_+ - \delta_w > 0$) or lowered ($\delta_+ - \delta_w < 0$) according as the top or the bottom layer has the large depth. Finally, in the limit $\rho_2 \to 0$, the formula (3.2) agrees with that for surface waves obtained by Benjamin (1970) and Whitham (1962).

The results of §3.1 have also been obtained by the use of ordinary flow force and energy methods, which are valid in steady internal-wave problems and have been applied several times before, e.g. by Benjamin (1966).

3.2. *Calculation of the upstream influence*

In §3.1 we found the result (3.2) for the change $\delta_+ - \delta_w$ in mean level behind the obstacle relative to the level in front. This was derived from an analysis of the steady flow prevailing near the body. To consider upstream influence, we must investigate the whole flow system.

We calculate the upstream influence δ_+ as follows: equation (2.15*a*), integrated between A and D, gives

$$(c_0 - c)[\rho\mathscr{U}_+] + (c - c_g)[\rho\mathscr{U}_w] + (c_0 + c_g)[\rho\mathscr{U}_-] = 0. \qquad (3.4a)$$

For the case of a free-surface flow ($\rho_2 = 0$) this equation states that the spatial average of the horizontal velocity over the whole system is zero. We eliminate $\mathscr{U}_1$ by using the mass conservation equations,

$$d_1 U_{1w-} = d_1\left(\mathscr{U}_{1w} + \frac{E_w f_1}{\rho_1 c_1 d_1}\right) = (c_0 - c)\delta_+ + c\delta_w,$$

$$d_1 U_{1+} = d_1\mathscr{U}_{1+} = c_0\delta_+,$$

$$d_1 U_{1-} = d_1\mathscr{U}_{1-} = -c_0\delta_-;$$

and similarly $\mathscr{U}_2$. These give

$$(c_0 - c)(c_0 + c - c_g)\delta_+ + c(c - c_g)\delta_w - c_0(c + c_g)\delta_- = \frac{c_0^2(c - c_g)E_w}{g(\rho_1 - \rho_2)c}\left[\frac{f}{d}\right]. \qquad (3.4b)$$

Finally, the overall condition of mass conservation is

$$(c_0-c)\,\delta_+ + (c-c_g)\,\delta_w + (c_0+c_g)\,\delta_- = 0. \tag{3.5}$$

Equations (3.2), (3.4*b*) and (3.5) are a set of three linear equations for the three unknowns $\delta_+, \delta_w, \delta_-$. These may be solved for δ_+:

$$\delta_+ = \frac{E_w c_0(c-c_g)}{2g(\rho_1-\rho_2)\,(c_0-c_g)}\left\{\frac{c_0}{c}\left[\frac{f_1}{d}\right] - \frac{1}{2}\left[\frac{\partial f}{\partial d}\right]\right\}. \tag{3.6}$$

The condition for no upstream influence, $\delta_+ = 0$, is thus

$$c_0[f/d] = \tfrac{1}{2}c[\partial f/\partial d].$$

For the Boussinesq approximation with $d_1 = d_2$ there is no upstream influence ($\delta_+ = 0$). In §3.1 it was demonstrated that, in this particular case, the mean levels upstream and in the wave train are the same: $\delta_+ - \delta_w = 0$; moreover, by the mass-conservation equation (3.5), $\delta_- = 0$. (We remark that this does not contradict Benjamin's (1970) argument because here there is impulse in the wave train.) If the Boussinesq approximation is not made, it is possible to have no upstream influence ($\delta_+ = 0$) and yet a change in the mean height of the wave train ($\delta_+ - \delta_w \neq 0$).

One can again consider various limits. Making use of (2.11) in the limit $\rho_2 \to 0$, we recover Benjamin's results for the free-surface problem: a positive surge propagating upstream ($\delta_+ > 0$), a decrease in level behind the body ($\delta_+ - \delta_w > 0$ with $\delta_w < 0$), and a negative surge propagating downstream ($\delta_- < 0$). Benjamin (1970, figure 2) has given a graph of the quantities $\delta_+ d_1/a^2$, $\delta_w d_1/a^2$, $\delta_- d_1/a^2$ as functions of the Froude number F, where $F^2 = c^2/c_0^2 = (\tanh kd_1)/kd_1$. For long waves, $k \to 0$, the surge is positive (as in the free-surface problem) or negative, according as

$$\rho_1/\rho_2 \gtrless d_1^2/d_2^2.$$

A different limit is obtained if one of the layers is very much deeper than the other: when the upper layer is much deeper than the lower, the forward surge is positive; if the upper layer is much more shallow than the lower, the forward surge is negative.

4. Results: shear across the interface

The model described in this section is admittedly artificial. The linearized initial-value problem is not well set, because of the Kelvin–Helmholtz instability, and there is no reason to suppose that the non-linear problem is better in this respect. However, the results below may be suggestive for the treatment of continuous stratification. Choose axes fixed with respect to the body; let the undisturbed depths d_1, d_2, and undisturbed velocities V_1, V_2, define a subcritical flow. We suppose that, after a large time, a steady wave train is set up on one side of the obstacle, which will be called the 'downstream' side; and that as before, the second-order mean height changes are propagated at the long wave speeds c_+, c_- and at the group velocity c_g. We quote the following two results on the vanishing of the upstream influence δ_+. (i) If the layer depths are equal and

$\rho_1 V_1^2 = \rho_2 V_2^2$, then $\delta_+ = 0$. (The analogy with the Long model is suggestive.) (ii) For a given pair of densities and basic velocities V_1, V_2 (which allow subcritical flows), one can always find a ratio d_2/d_1 of layer depths for which the upstream influence vanishes.

5. Conclusions

The main qualitative result is as follows: when the bottom layer is the more shallow, the case of waves at an interface is very similar to the case of surface waves. But when the top layer is the more shallow, the mean height changes in the present situation of interfacial waves are opposite to those in the case of surface waves (figure 3). In view of this change of sign, it is not surprising that there are special instances in which there is no upstream influence. Such a continuity argument suggests that there will always be a ratio of layer depths for which there is no upstream influence to all orders in the amplitude expansion.

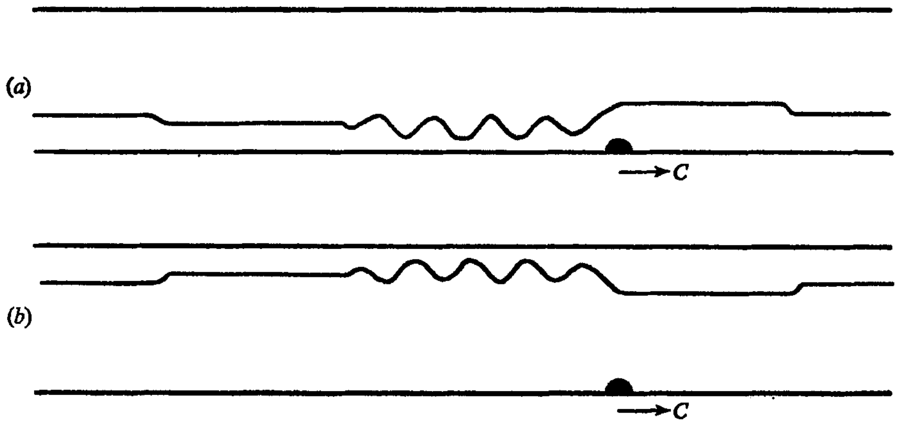

FIGURE 3. Height changes when (*a*) the lower layer or (*b*) the upper layer, is the shallower.

There is an alternative approach to questions of upstream influence; namely explicit expansion in a small parameter and the solution of a sequence of initial-value problems for the successive terms of the expansion. McIntyre (1971) has used such a procedure to obtain the upstream influence to second order for uniformly stratified and rotating flows. If results reached by this method agree with those reached by the method used in § 3, in such cases where both methods can be applied, it will be desirable to put the entire perturbation scheme on a firmer basis; it will also be desirable to determine whether upstream influence exists at very large time.

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